(IJISE) 2022, Vol. No. 16, Jul-Dec **e-ISSN: 2454-6402, p-ISSN: 2454-812X**

COMMON FIXED POINT THEOREMS FOR TWO SELFMAPS OF A COMPACT S−METRIC SPACE

Upender. S

Associate Professor of Mathematics, Tara Government College (Autonomous), Sangareddy -502001, India

ABSTRACT

The purpose of this paper is to prove a common fixed point theorem for two selfmaps on a S–metric space and deduce a common fixed point theorem for two selfmaps on a compact S–metric space. Further we show that a common fixed point theorem for two selfmaps of a metric space prove by Brian Fisher ([5]) is a particular case of our theorem.

Mathematics Subject Classification*: 47H10, 54H25.*

Key Words*: S-metric space; Compatible; Fixed point theorem.*

1. INTRODUCTION AND PRELIMINARIES

Fixed point theory has great implications in the field of analysis. Fixed point theory is used to find solutions of different mathematical problems like integral equations, differential equations, optimization problems, convex minimization problems, image recovery, signal processing etc. Several mathematicians studied fixed point results over different spaces as metric space, Banach space, Reflexive space, Hilbert space and many more. One of the most important and fruitful result in metric space was given by Banach called "Banach Contraction Principle". This principle was generalized and its several variants were studied by mathematicians over different spaces.

On the other hand, some authors are interested and have tried to give generalizations of metric spaces in different ways. In 1963 Gahler [6] gave the concepts of 2- metric space further in 1992 Dhage [2] modified the concept of 2- metric space and introduced the concepts of D-metric space also proved fixed point theorems for selfmaps of such spaces. Later researchers have made a significant contribution to fixed point of D- metric spaces in [1], [3], and [4]. Unfortunately almost all the fixed point theorems proved on D-metric spaces are not valid in view of papers [7], [8] and [9]. Sedghi et al. [10] modified the concepts of D- metric space and introduced the concepts of D*- metric space also proved a common fixed point theorems in D*- metric space.

Recently, Sedghi et al [11] introduced the concept of S- metric space which is different from other space and proved fixed point theorems in S-metric space. They also gives some examples of S- metric spaces which shows that S- metric space is different from other spaces. In fact they gives following concepts of S- metric space.

Definition 1.1([11]): Let X be a non-empty set. An S-metric space on X is a function $S: X^3 \to [0, \infty)$ that satisfies the following conditions, for each x, y, z, a ∈ X

(i) $S(x, y, z) \geq 0$

(ii) $S(x, y, z) = 0$ if and only if $x = y = z$.

(iii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$

The pair (X, S) is called an **S–metric space**.

Immediate examples of such S-metric spaces are:

Example 1.2: Let ℝ be the real line. Then $S(x, y, z) = |x - y| + |y - z| + |z - x|$ for each x, y, z ∈ℝ is an S-metric on ℝ. This S-metric is called the usual S-metric on ℝ.

Example 1.3: Let $X = \mathbb{R}^2$, d be the ordinary metric on X.

Put $S(x, y, z) = d(x, y) + d(y, z) + d(z, x)$ is an S- metric on X. If we connect the points x, y, z by a line, we have a triangle and if we choose a point a mediating this triangle then the inequality $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ holds. In fact

$$\begin{aligned} S(x, y, z) &= d(x, y) + d(y, z) + d(z, x) \\ &\leq d(x, a) + d(a, y) + d(y, a) + d(a, z) + d(z, a) + d(a, x) \\ &= S(x, x, a) + S(y, y, a) + S(z, z, a) \end{aligned}$$

Example 1. 4: Let $X = \mathbb{R}^n$ and $\|\ .\ \|$ a norm on X, then $S(x, y, z) = \|x - z\| + \|y - z\|$ is an S-metric on X.

Remark 1. 5: it is easy to see that every D*-metric is S-metric, but in general the converse is not true, see the following example.

Example 1. 6: Let $X = \mathbb{R}^n$ and $\|\ .\ \|$ a norm on X, then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is an S-metric on X, but it is not D*-metric because it is not symmetric.

Lemma 1. 7: In an S–metric space, we have $S(x, x, y) = S(y, y, x)$.

Proof: By the third condition of S-metric, we get

$S(x, x, y) \leq S(x, x, x) + S(x, x, x) + S(y, y, x) = S(y, y, x)$…… (1)

and similarly

$S(y, y, x) \leq S(y, y, y) + S(y, y, y) + S(x, x, y) = S(x, x, y)$……(2)

Hence, by (1) and (2), we obtain $S(x, x, y) = S(y, y, x)$.

Definition1.8: Let (X, S) be an S-metric space. For x ∈X and r > 0, we define the open ball $B_S(x, r)$ and closed ball $B_S[x, r]$ with a center x and a radius r as follows

$$B_S(x, r) = \{y \in X; S(x, y, y) < r\}$$

$$B_S[x, r] = \{y \in X; S(x, y, y) \leq r\}$$

For example, Let X =ℝ. Denote $S(x, y, z) = | y + z - 2x | + | y - z |$ for all x, y, z ∈ℝ. Therefore

$$\begin{aligned} B_S(1, 2) &= \{y \in \mathbb{R} ; S(y, y, 1) < 2\} \\ &= \{y \in \mathbb{R} ; | y - 1| < 1\} = (0, 2). \end{aligned}$$

(IJISE) 2022, Vol. No. 16, Jul-Dec **e-ISSN: 2454-6402, p-ISSN: 2454-812X**

Definition 1.9: Let (X, S) be an S–metric space and $A \subset X$.

(1)If for every $x \in A$, there is a $r > 0$ such that $B_S(x, r) \subset A$, then the subset A called an **open subset** of X

(2) If there is a $r > 0$ such that $S(x, x, y) < r$ for all $x, y \in A$ then A is said to be **S–bounded.**

(3) A sequence $\{x_n\}$ in X **converge to x** if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is for each $\in > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n, x_n, x) < \in$ and we denote this by $\lim_{n \to \infty} x_n = x$

(4) A sequence $\{x_n\}$ in X is called a **Cauchy sequence** if for each $\in > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \in$ for each $m, n \geq n_0$

(5) The S–metric space (X, S) is said to be **complete** if every Cauchy sequence is convergent sequence.

(6) Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $r > 0$ such that $B_S(x, r) \subset A$. Then τ is a topology on X (induced by the S-metric S).

(7) If (X, τ) is a compact topological space we shall call (X, S) is a **compact** S–metric space.

Lemma1. 10([11]): Let (X, S) be an S-metric space. If $r > 0$ and $x \in X$, then the open ball $B_S(x, r)$ is an open subset of X.

Lemma1. 11([11]**):** Let (X, S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x, then x is unique.

Lemma1. 12([11]**):** Let (X, S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x, then $\{x_n\}$ is a Cauchy sequence.

Lemma1. 13([11]**):** Let (X, S) be an S-metric space. If there exists sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$, then $\lim_{n \to \infty} S(x_n, x_n, y_n) = S(x, x, y)$.

Lemma1. 14: Let (X, d) be a metric space. Then we have

1. $S_d(x, y, z) = d(x, y) + d(y, z) + d(z, x)$ for all $x, y, z \in X$ is an S-metric on X
2. $x_n \to x$ in (X, d) if and only if $X_n \to x$ in (X, S_d)
3. $\{x_n\}$ is a Cauchy sequence in (X, d) if and only if $\{x_n\}$ is a Cauchy sequence in (X, S_d)
4. (X, d) is complete if and only if (X, S_d) is complete

Proof: **(1)** See [Example (3), Page 260]

(2) $x_n \to x$ in (X, d) if and only if $d(x_n, x) \to 0$, if and only if $S_d(x_n, x_n, x) = 3d(x_n, x) \to 0$ that is, $x_n \to x$ in (X, S_d)

(3) $\{x_n\}$is a Cauchy in (X, d) if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$, if and only if $S_d(x_n, x_n, x_m) = 3d(x_n, x_m) \to 0$ $n, m \to \infty$, that is, $\{x_n\}$ is Cauchy in (X, S_d)

(4) It is a direct consequence of (2) and (3)

Notation: For any selfmap T of X, we denote T(x) by Tx.
If P and Q are selfmaps of a set X, then any $z \in X$ such that Pz = Qz = z is called a **common fixed point** of P and Q.
Two selfmaps P and Q of X are said to be **commutative** if PQ = QP where PQ is their composition PoQ defined by (PoQ) x = PQx for all $x \in X$.

Definition 1.15: Suppose P and Q are selfmaps of a S–metric space (X, S) satisfying the condition $Q(X) \subseteq P(X)$. Then for any $x_0 \in X$, $Qx_0 \in Q(X)$ and hence $Qx_0 \in P(X)$, so that there is a $x_1 \in X$ with $Qx_0 = Px_1$, since $Q(X) \subseteq P(X)$. Now $Qx_1 \in Q(X)$ and hence there is a $x_2 \in X$ with $Qx_2 \in Q(X) \subseteq P(X)$ so that $Qx_1 = Px_2$. Again $Qx_2 \in Q(X)$ and hence $Qx_2 \in P(X)$ with $Qx_2 = Px_3$. Thus repeating this process to each $x_0 \in X$, we get a sequence $\{x_n\}$ in X such that $Qx_n = Px_{n+1}$ for $n \geq 0$. We shall call this sequence as an **associated sequence of** x_0 **relative to the two selfmaps P and Q.** It may be noted that there may be more than one associated sequence for a point $x_0 \in X$ relative to selfmaps P and Q.

Let P and Q are selfmaps of a S-metric space (X, S) such that $Q(X) \subseteq P(X)$. For any $x_o \in X$, if $\{x_n\}$ is a sequence in X such that $Qx_n = Px_{n+1}$ for $n \geq 0$, then $\{x_n\}$ is called an **associated sequence** of x_0 relative to the two selfmaps P and Q.

Definition 1.16: A function $\varnothing: [0, \infty) \rightarrow [0, \infty)$ is said to be a **contractive modulus**, if $\varnothing(0) = 0$ and $\varnothing(t) < t$ for $t > 0$.

Definition 1.17: A real valued function Ø defined on $X \subseteq \mathbb{R}$ is said to be **upper semi continuous**, if $\lim_{n\to\infty} \sup \varnothing(t_n) \leq \varnothing(t)$ for every sequence $\{t_n\}$ in X with $t_n \rightarrow t$ as $n \rightarrow \infty$.

Definition 1.18: If P and Q are selfmaps of a S-metric space (X, S) such that for every sequence $\{x_n\}$ in X with $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qx_n = t$, we have
$\lim_{n\to\infty} S(PQx_n, QPx_n, QPx_n) = 0$, then we say that P and Q are **compatible.**

2. THE MAIN RESULTS:

2.1 Theorem. Suppose P and Q are selfmaps of a S–metric space (X, S) satisfying the conditions

(i) $Q(X) \subseteq P(X)$

(ii) $S(Qx, Qy, Qy) \leq \beta(x, y)$ for all $x, y \in X$,
where

(ii)′ $\beta(x, y) = \max\{S(Px, Py, Py), S(Px, Qx, Qx), S(Py, Qy, Qy), \frac{1}{2}[S(Px, Qy, Qy) + S(Py, Qx, Qx)]\}$

(iii) P and Q are continuous.

(iv) the pair (P, Q) is compatible,
and

(v) there is a point $x_0 \in X$ and an associated sequence $\{x_n\}$ of x_0 relative to the two selfmaps such that the sequences $\{Qx_n\}$ and $\{Px_n\}$ converge to some point $z \in X$. Further, if

(vi) there exists $(p, q) \in X^2$ such that $f(p, q) = \sup_{(x,y)\in X^2} f(x,y)$, where

(vi)′ $f(x, y) = \frac{S(Px,Qy,Qy)}{\beta(x,y)}$

then P and Q have a unique common fixed point $z \in X$.

Proof: First suppose that $\beta(x, y) > 0$ for all $x, y \in X$, so that $f(x, y)$ is well defined. Now by the inequality (ii), we find that $f(x, y) < 1$ for all $x, y \in X$. Hence if $c = f(p, q)$ then $c \leq 1$, so that $f(x, y) \leq c$ for all $x, y \in X$ and therefore $S(Qx, Qy, Py) \leq c\,\beta(x, y)$

From (v), we get

(2.1.1) $Px_{2n}, Qx_{2n}, Px_{2n+1}$ and $Qx_{2n+1} \to z$ as $n \to \infty$

Now, since P, Q are continuous, we have by (2.1.1)

$P^2x_{2n} \to Pz$, and $PQx_{2n+1} \to Pz$ as $n \to \infty$

Since the pair (P, Q) is compatible, we have, in view of (2.1.1) that

$\lim_{n\to\infty} S(PQx_{2n+1}, QPx_{2n+1}, QPx_{2n+1}) = 0$,

$QPx_{2n+1} \to Sz$ as $n \to \infty$

Also from (ii), we have

(2.1.2) $S(QPx_{2n+1}, Qx_{2n}, Qx_{2n}) \leq \beta(Sx_{2n+1}, x_{2n})$,

where $\beta(Px_{2n+1}, x_{2n}) = \max\{S(P^2x_{2n+1}, Px_{2n}, Px_{2n}), S(P^2x_{2n+1}, QPx_{2n}, QPx_{2n}), S(Px_{2n}, Qx_{2n}, Qx_{2n}),$

$\frac{1}{2}[S(P^2x_{2n+1}, Qx_{2n}, Qx_{2n}) + S(Px_{2n}, QPx_{2n+1}, QPx_{n+1})]\}$

which on letting n to ∞ and using the continuity of S gives

$\lim_{n\to\infty} \beta(Px_{2n+1}, Qx_{2n}) = \max\{S(Pz, z, z), S(Pz, Pz, Pz), S(z, z, z),$

$\frac{1}{2}[S(Pz, z, z) + S(Pz, z, z)]\}$

$= S(Pz, z, z)$

Therefore letting n to ∞ in (2.1.2), and using the above we get

(2. 1. 3) $S(Pz, z, z) \leq S(Pz, z, z)$.

Now, if $Pz \neq z$, then $S(Pz, z, z) > 0$ and by the definition, we get $S(Pz, z, z) < S(Pz, z, z)$, contradicting (2.1.3)

Thus we have $Pz = z$.

Now again from (ii) we have

(2.1.4) $S(Qz, Qx_{2n}, Qx_{2n}) \leq \beta(z, x_{2n})$

where $\beta(z, x_{2n}) = \max\{S(Pz, Px_{2n}, Px_{2n}), S(Pz, Qz, Qz), S(Px_{2n}, Qx_{2n}, Qx_{2n}),$

$\frac{1}{2}[S(Pz, Qx_{2n}, Qx_{2n}) + S(Px_{2n}, Qz, Qz)]\}$, in

which on letting n to ∞ and using $Pz = z$, the continuity of S and the condition (v), we get

$\lim_{n\to\infty} \beta(z, x_{2n}) = \max\{S(Pz, z, z), S(z, Qz, Qz), S(z, z, z),$

$\frac{1}{2}[S(Pz, z, z) + S(Pz, z, z)]\}$

$= S(z, Qz, Qz)$

Again letting n to ∞in (2.1.4), and using the above we get $S(Qz, z, z) \leq S(Qz, z, z)$

and this will be contradiction if $Qz \neq z$, therefore $Qz = z$. Thus z is a common fixed point of P and Q.

To prove that z is unique, if possible suppose that z' is another common fixed point of P and Q. Then from (ii), we have

(2.1.5) $S(z, z', z') = S(Qz, Qz', Qz') \leq \beta(z, z')$

where $\beta(z,, z') = \max \{S(Pz, z', z'), S(z, Qz, Qz), S(Pz', Qz', Qz'), \frac{1}{2}[S(Pz, z', z') + S(Pz', Qz, Qz)]\}$

$= S(z, z', z')$

so that (2.1.5) gives $S(z, z, z') \leq S(z, z, z')$ and this will give a contradiction if $z \neq z'$. Therefore $z = z'$. Thus z is the unique common fixed point of P and Q.

Now suppose that $\beta(x', y') = 0$ for some $x', y' \in X$. Then

$\max\{S(Px', Py', Py'), S(Px', Qx', Qx'), S(Py', Qy', Qy'), \frac{1}{2}[S(Px', Qy', Qy') + S(Py', Qx', Qx')]\} = 0,$

which implies

(2.1.6) $Px' = Qx' = Py' = Qy'$

Then $PQx' = P(Px') = P^2x' = PPy'$. Since the pair (P, Q) is compatible

(2.1.7) $\lim_{n\to\infty} S(PQx_n, QPx_n, QPx_n) = 0$

whenever $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Let $x_n = x'$, then $Px_n \to Px'$, $Qx_n \to Qx'$ as $n \to \infty$.

Therefore (2.1.7) and the continuity of S give $S(PQx', QPx', QPx') = 0$, which implies

(2.1.8) $PQx' = QPx' = Q^2x'$

If $Qx' \neq Q^2x'$, then from (ii) we have

(2.1.9) $S(Qx', Q^2x', Q^2x') < \beta(x', Qx')$

But by (2.1.6) and (2.1.8), we have

$\beta(x', Qx') = \max \{S(Px', PQx', PQx'), S(Px', Q^2 x', Q^2x'), S(PQx', Q^2x', Q^2x'), \frac{1}{2} [S(Px', Q^2x', Q^2x') + S(PQx', Qx', Qx')]\}$

$= S(Qx', Q^2x', Q^2x')$

This contradicts (2.1.9) if $Qx' \neq Q^2x'$.

Therefore $Qx' = Q^2x'$. Now $Qx' = Q^2x' = Q(Qx')$, showing that $Qx' = z$ is a fixed point of Q.

Further $Pz = PQx' = QPx' = Q^2x' = Qz = z$.

Therefore z is also a fixed point of P. Hence z is a common fixed point of P and Q.

Now we prove the uniqueness of the common fixed point. If possible assume that z' is another common fixed point of P and Q. If $z \neq z'$, then from (ii) we have

$S(z, z', z') = S(Qz, Qz', Qz') < \beta(z, z')$

where $\beta(z,, z') = \max \{S(Pz, Pz', Pz'), S(Pz, Qz, Qz), S(Pz', Qz', Qz'), \frac{1}{2} [S(Pz, Qz', Qz') + S(Pz', Qz, Qz)]\}$

= S(z, z', z'),

This impossibility shows z = z'.

Hence z is the unique common fixed point of P and Q.

As a consequence of Theorem 2.1, we have the following

2.2 Corollary: Suppose (X, S) is a S-metric space satisfying conditions (i), (ii), (iii) and (iv) of Theorem 2.1. Further, if (X, S) is compact. Then P and Q have unique common fixed point z.

Proof: Since (X, S) is a compact S-metric space, it is complete and therefore for each $x_0 \in X$ and for any associated sequence $\{x_n\}$ of x_0 relative to two selfmaps such that the sequences $\{Px_n\}$ and $\{Qx_n\}$ converge to some $z \in X$ and hence condition (v) of Theorem 2.1 holds . Also, if (X, S) is compact S-metric space, then f(x, y) is continuous function on the compact S-metric space X^2. Therefore we can find (p, q)$\in X^2$ such that f(p, q) = $\sup_{(x,y)\in X^2} f(x,y)$, proving the condition (vi) of the Theorem 2.1. Hence by Theorem 2.1, the corollary follows.

2.3 Corollary ([5]): Suppose P and Q are two selfmaps of metric space (X, d) such that

(i) $Q(X) \subseteq P(X)$

(ii) $d(Qx, Qy) < \alpha(x, y)$ for all $x, y \in X$.

where

(ii)' $\alpha(x, y) = \max \{d(Px, Py), d(Px, Qx), d(Py, Qy), d(Px, Qy), d(Py, Qx)\}$

(i) P and Q are continuous,

and

(iv) PQ = QP, further if

(v) X is compact.

Then P and Q have a unique common fixed point.

Proof: Given (X, d) is a metric space satisfying condition (i) to (v) of the corollary.

If $S(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$, then (X, S) is a S-metric space and

$S(x, y, x) = d(x, y)$ Therefore (ii) can be written as $S(Qx, Qy, Qy) < \alpha(x, y)$ for all $x, y \in X$, where $\alpha(x, y) = \max \{S(Px,Py,Py), S(Px,Px,Px), S(Py,Qy,Qy), S(Px,Qy,Qy), S(Py, Qx,Qx)\}$ which is the same as condition (ii) of Theorem 2.1. Also since (X, d) is complete, we have (X, S) is complete, by Corollary 1.13.

Now P and Q are selfmaps on (X, S) satisfying conditions of Corollary 2.2 and hence the corollary follows.

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